RESEARCH OF NONSTATIONARY LAMINAR PLANE-PARALLEL PRESSURE FLOW

A.SARUKHANYAN, A.MANUKYAN

State University of Architecture and Construction of Erevan

Nonstationary flat-parallel pressure flow of viscous liquid under arbitrary initial and boundary conditions are reviewed.
Regularities of instantaneous speed distribution on the effective cross-section for a general case when initial speed distribution and pressure difference are given in form of arbitrary function are obtained. The obtained regularities of the instantaneous speeds allow to calculate momentum and kinetic energy factors, power losses in the unbalanced flow, determine the regularity of the profile speed variation (deformation) in time, obtain the character of change of momentum β and kinetic energy α factors.

Key words: Viscous fluid, boundary conditions, laminar motion, instantaneous speeds, nonstationary flow, quantity of flow.

Research of nonstationary processes in pressure systems is important to properly understand their essence and select design diagrams in constructing various systems and devices widely used in the industry. Plane-parallel model of the pressure flow is essential for studying the flow pattern characteristic to a lot of systems and pressure lines of the automated complex devices [1,2].

Let's review the plane-parallel pressure flow of viscous fluid between two unlimited parallel fixed walls the distance between which is *2h* (figure). Let's review the fluid flow in Cartesian coordinate system *oxyz*, commencement of which will be placed in a centre between the plates, and direct the coordinate *x* along the flow axis. Assuming that fluid flows just along the axis *ox*, $\mathbf{u_x \neq 0,\ u_y = u_z = 0}$.

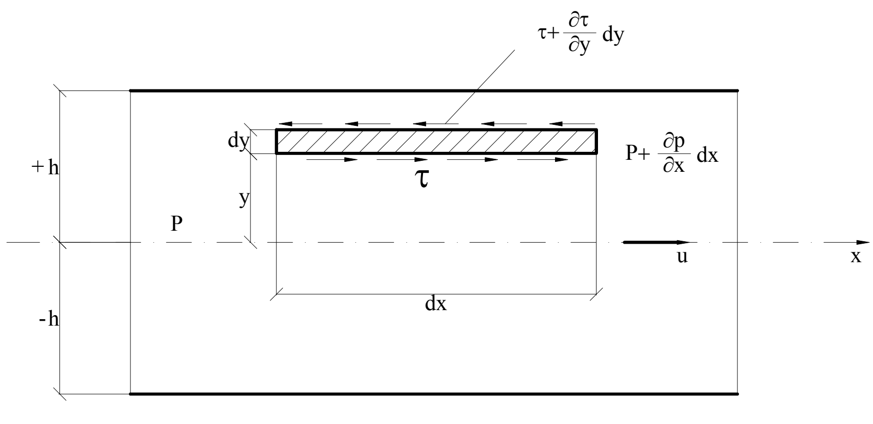

Figure.

Without considering initial area or the area of flow generation where hydromechanic parameters change again along the flow, let's review conventionalized flow area where hydraulic parameters change just along the effective cross-section. Under such assumptions, Navier- Stokes equations and continuities, without taking into the account the body forces, will have the following form:

$$\frac{\partial u_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u_x}{\partial y^2}, \qquad \frac{\partial u_x}{\partial x} = 0; \tag{1}$$

$$\frac{\partial p}{\partial y} = 0; \tag{2}$$

$$\frac{\partial p}{\partial z} = 0. \tag{3}$$

It is obvious from the last two equations that pressure in the effective cross-section does not depend on the coordinates and depends just on time, i.e.

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = f(t). \tag{4}$$

Assuming that distribution of speeds along the effective cross-section at an initial time instant has an arbitrary shape depending on coordinate *y*, i.e. $u_x = \varphi(y)$, at t=0 the research of the nonstationary plane-parallel pressure flow leads to the integration of differential equation

$$\frac{\partial u_x}{\partial t} = f(t) + \nu\frac{\partial^2 u_x}{\partial y^2} \tag{5}$$

In following initial and boundary conditions:

$$u_x(y, t) = 0, \text{ in } y = \pm h, \quad t > 0; \tag{6}$$

$$u_x(y, t) = \varphi(y) \text{ in } t = 0, \quad (-h < y < +h). \tag{7}$$

Let's introduce non-dimensional coordinates and parameters

$$u_x = u_\infty \cdot u_0; \quad x = h \cdot x_0; \quad y = h \cdot y_0; \quad t = \frac{h}{u_\infty} \cdot t_0, \tag{8}$$

where $p = \rho \cdot u_\infty^2 \cdot p_0$; $\quad u_\infty = \lim\limits_{t\to\infty} u_x(0,t)$.

Then the equation (1) will convert in a following way

$$\frac{\partial(u_\infty u_0)}{\partial\left(\frac{h}{u_\infty}\cdot t_0\right)} = -\frac{1}{\rho}\cdot\frac{\rho u_\infty^2}{h}\frac{\partial p_0}{\partial x_0} + \nu\frac{\partial}{h\partial y_0}\cdot\frac{\partial(u_\infty u_0)}{h\partial y_0}.$$

In simplifying the last equation, will obtain:

$$\frac{\partial u_0}{\partial t_0} = -\frac{\partial p_0}{\partial x_0} + \frac{1}{Re}\frac{\partial^2 u_0}{\partial y_0^2}, \tag{9}$$

where $Re = \frac{u_\infty h}{\nu}$ - Reynolds number.

Let's also present the initial and boundary conditions in non-dimensional form:

1) $$u_x(y,t) = u_x(h \cdot y_0, \frac{h}{u_\infty} t_0)\Big|_{y_0=\pm 1} = 0 \quad \to$$

$$\to \quad \frac{u_\infty u_x(h \cdot y_0, \frac{h}{u_\infty} t_0)}{u_\infty}\Bigg|_{y_0=\pm 1} = u_\infty u_0(h \cdot y_0, \frac{h}{u_\infty} t_0)\Big|_{y_0=\pm 1} \Rightarrow u_0(y_0, t_0)\big|_{y_0=\pm 1} = 0;$$

2) $$u_x(y,0) = \varphi(y) \Rightarrow \frac{u_\infty u_x(y,0)}{u_\infty} = \frac{u_\infty \varphi(y)}{u_\infty} \Rightarrow$$

$$\Rightarrow u_0(y_0,0) = \varphi_0(y_0), \quad \text{при } t_0 = 0, \quad -1 \le y_0 \le +1 \quad \left(\varphi_0(y_0) = \frac{\varphi(hy)}{u_\infty}\right),$$

3) $$f(t) = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \to -\frac{\rho u_\infty^2 \partial p_0}{\rho \partial h x_0} = -\frac{u_\infty^2}{h}\frac{\partial p_0}{\partial x_0} \Rightarrow \frac{\partial p_0}{\partial x} = f_0(t_0) \quad \left(f_0(t_0) = \frac{hf\left(\frac{h}{u_\infty} t_0\right)}{u_\infty^2} \right).$$

Dropping zero indexes in non-dimensional values will obtain the boundary value problem. Let's integrate the differential equation

$$\frac{\partial u}{\partial t} = f(t) + \frac{1}{Re}\frac{\partial^2 u}{\partial y^2} \tag{10}$$

under the following conditions:

$$u(y,\, t) = 0, \text{ in } \quad y = \pm 1, \quad t > 0; \tag{11}$$

$$u(y,\, t) = \varphi(y) \text{ in } \quad t = 0, \quad (-1 < y < +1). \tag{12}$$

In stationary flow , distribution of the speeds along the effective cross-section has the form of square parabola

$$u_{ст} = \frac{Re \cdot p}{2}\left(1 - y^2\right) = u_\infty\left(1 - y^2\right), \tag{13}$$

where $u_\infty = \frac{Re \cdot p}{2}$ - maximum speed at the flow centre.

Average speed of the effective cross-section will be:

$$V = \frac{Re \cdot p}{3} = \frac{2}{3} u_\infty . \tag{14}$$

Coefficients of momentum β and maldistribution of speeds α will be written in a following form:

$$\beta = \frac{1}{2}\frac{\int_{-1}^{+1}\left(\frac{Re \cdot p}{2}\left(1 - y^2\right)^2\right)dy}{\left(\frac{Re \cdot p}{3}\right)^2} = 1.2; \tag{15}$$

$$\alpha = \frac{1}{2}\frac{\int_{-1}^{+1}\left(\left(\frac{Re \cdot p}{2}\right)^{3.}\left(1 - y^2\right)^3\right)dy}{\left(\frac{Re \cdot p}{3}\right)^3} = \frac{54}{35} . \tag{16}$$

Solution of the equation (10) under the conditions (11), (12) will be sought in form of a sum [4]

$$u_x(y,t)=u_1(y,t)+u_2(y,t)\ , \tag{17}$$

where $u_1(y,t)$ is the solution of heterogeneous differential equation:

$$\frac{\partial u_1}{\partial t}=\frac{1}{Re}\frac{\partial^2 u_1}{\partial y^2} \tag{18}$$

Under heterogeneous edge conditions:

$$u_1(y,\ t)=0,\ \text{ in } y=\pm 1,\quad t>0; \tag{19}$$

$$u_1(y,\ t)=\varphi(y),\ \text{ in } t=0,\ \ (-1<y<+1); \tag{20}$$

Function $u_1(y,t)$ includes speed deviation from initial distribution $u_2(y,t)$ along the effective cross-section and deviation from pressure differential. Solution of the equation (18) will be sought in form of a sum of infinite series:

$$u_1(y,t)=\sum_{k=1}^{\infty}C_k\cos(\frac{(2k-1)}{2}\pi y)\cdot\exp\left(-\frac{\pi^2(2k-1)^2}{4\,Re}t\right), \tag{24}$$

where C_k – constant coefficients the values of which are determined according to the initial condition of the problem (20). Having

$$u_1(y,0)=\varphi(y)=\sum_{k=1}^{\infty}C_k\cos(\frac{(2k-1)}{2}\pi y). \tag{25}$$

for determining C_k coefficient values, let's multiply both parts of the equalities (25) by $\cos\left(\frac{(2k-1}{2}\pi y\right)$ and prioritize at an interval (0;+1). Taking into consideration, that:

$$\int_0^{+1}\cos(\frac{(2k-1)}{2}\pi y)\cdot\cos(\frac{(2k-1)}{2}\pi y)dy=\begin{cases}0, & \text{при } k\neq n;\\ \frac{1}{2}, & \text{при } k=n,\end{cases} \tag{26}$$

will obtain

$$C_k=2\int_0^{+1}\varphi(y)\cdot\cos(\frac{2k-1}{2}\pi y)dy\ . \tag{27}$$

Solution of the heterogeneous equation (21) will be sought in form of a sum according to the eigenfunction of the problems, i.e.

$$u_2(y,t)=\sum_{k=1}^{\infty}b_k(t)\cdot\cos(\frac{2k-1}{2}\pi y)\exp\left(-\frac{\pi^2(2k-1)^2}{4\,Re}t\right)\ , \tag{28}$$

where $b_k(t)$ - variable coefficient determined under the initial conditions of the problem (23), i.e.

$$b_k(t)\big|_{t=0}=0\ . \tag{29}$$

where

$$\gamma_k(t)=2\int_0^1 f(t)\cos\ (\frac{2k-1}{2}\pi y)\,dy\ . \tag{32}$$

Integrating the last expression, will obtain:

$$\gamma_k(t)=\frac{4f(t)}{(2k-1)\pi}\cdot\sin\frac{(2k-1)\pi}{2}y\Big|_0^{+1}=\frac{4f(t)}{(2k-1)\pi}\cdot\sin\frac{(2k-1)\pi}{2}=\frac{(-1)^{k+1}4f(t)}{(2k-1)\pi}\ , \tag{33}$$

following to which the coefficients $\mathbf{b}_k(\mathbf{t})$ are calculated from the linear differential equation

$$\mathbf{b}'_k(\mathbf{t})\cdot\mathbf{exp}\left(-\frac{\pi^2(\mathbf{2k-1})^2}{\mathbf{4\,Re}}\mathbf{t}\right)=\frac{(-\mathbf{1})^{\mathbf{k+1}}\mathbf{4f}(\mathbf{t})}{(\mathbf{2k-1})\pi}. \tag{34}$$

Solution of the equation (34) will be:

$$\mathbf{b}_k(\mathbf{t})=-\frac{\mathbf{4}(-\mathbf{1})^{\mathbf{k+1}}\mathbf{F}(\mathbf{t})}{\pi(\mathbf{2k-1})}; \tag{35}$$

where

$$\mathbf{F}(\mathbf{t})=\int_0^t\mathbf{f}(\mathbf{u})\cdot\mathbf{exp}\left(\frac{\pi^2(\mathbf{2k-1})^2}{\mathbf{4\,Re}}\mathbf{u}\right)\mathbf{du}. \tag{36}$$

So, in the plane-parallel nonstationary laminar flow in the arbitrary distribution of the speeds and in the pressure differential, instantaneous speeds along the effective scrods-section are distributed according to the law

$$\mathbf{u}_x(\mathbf{y},\mathbf{t})=\sum_{k=1}^{\infty}\left[\mathbf{C}_k+\frac{(-\mathbf{1})^{\mathbf{k+1}}\mathbf{4F}(\mathbf{t})}{(\mathbf{2k-1})\pi}\right]\cdot\mathbf{cos}\left[\frac{(\mathbf{2k-1})}{\mathbf{2}}\pi\mathbf{y}\right]\cdot\mathbf{exp}\left(-\frac{\pi^2(\mathbf{2k-1})^2}{\mathbf{4\,Re}}\mathbf{t}\right). \tag{37}$$

It should be noted, that the obtained solution applies to the general case of the problem from where it is possible to get specific solutions of a lot of practical problems.

Shearing stresses in laminar flow are specified according to Newton law [5,6]

$$\tau=\pm\mu\frac{\partial\mathbf{u}}{\partial\mathbf{y}}, \tag{38}$$

according to which the distribution of the shearing stresses along the effective cross-section in the nonstationary plane-parallel flow will be:

$$\tau=\mu\sum_{k=1}^{\infty}\pi\frac{(\mathbf{2k-1})}{\mathbf{2}}\left[\mathbf{C}_k+\frac{(-\mathbf{1})^{\mathbf{k+1}}\mathbf{4F}(\mathbf{t})}{(\mathbf{2k-1})\pi}\right]\cdot\mathbf{sin}\left[\pi\frac{(\mathbf{2k-1})}{\mathbf{2}}\mathbf{y}\right]\cdot\mathbf{exp}\left(-\frac{\pi^2(\mathbf{2k-1})^2}{\mathbf{4\,Re}}\mathbf{t}\right). \tag{39}$$

$$(-\mathbf{1}\le\mathbf{y}\le\mathbf{1},\quad\mathbf{t}>\mathbf{0}).$$

Average instantaneous speed of the effective cross-section:

$$\mathbf{V}=\int_0^1\mathbf{u}_x\mathbf{dy}=\frac{\mathbf{2}}{\pi}\sum_{k=1}^{\infty}\frac{\mathbf{1}}{\mathbf{2k-1}}\left[\mathbf{C}_k+\frac{(-\mathbf{1})^{\mathbf{k+1}}\mathbf{4F}(\mathbf{t})}{(\mathbf{2k-1})\pi}\right]\cdot\mathbf{exp}\left(-\frac{\pi^2(\mathbf{2k-1})^2}{\mathbf{4\,Re}}\mathbf{t}\right)\cdot\mathbf{sin}\,(\pi\frac{(\mathbf{2k-1})}{\mathbf{2}}\mathbf{y})\Bigg|_{y=0}^{y=1}=$$

$$=\frac{\mathbf{2}}{\pi}\sum_{k=1}^{\infty}\frac{(-\mathbf{1})^{\mathbf{k+1}}}{(\mathbf{2k-1})}\left[\frac{(-\mathbf{1})^{\mathbf{k+1}}\mathbf{4F}(\mathbf{t})}{(\mathbf{2k-1})\pi}+\mathbf{C}_k\right]\cdot\mathbf{exp}\left(-\frac{\pi^2(\mathbf{2k-1})^2}{\mathbf{4\,Re}}\mathbf{t}\right).$$

therefore,

$$\mathbf{V}=\frac{\mathbf{2}}{\pi}\sum_{k=1}^{\infty}\frac{(-\mathbf{1})^{\mathbf{k+1}}}{(\mathbf{2k-1})}\left[\frac{(-\mathbf{1})^{\mathbf{k+1}}\mathbf{4f}(\mathbf{t})}{(\mathbf{2k-1})\pi}+\mathbf{C}_k\right]\cdot\mathbf{exp}\left(-\frac{\pi^2(\mathbf{2k-1})^2}{\mathbf{4\,Re}}\mathbf{t}\right);\qquad\mathbf{0}\le\mathbf{t}<\infty. \tag{40}$$

Regularity of flow quantity coefficient change is specified according to the equations (37) and (40). Will obtain:

$$\beta = \frac{\int_{-1}^{+1} u^2 dy}{2V^2} = \frac{1}{V^2}\int_0^1 u_x^2 dy\ . \tag{41}$$

For calculating the integral in numerator, let's apply Percival equality properties [3], which is fair for orthogonal function system. Therefore, let's bring the system of eigenfunctions of the problem to normalized orthogonal system. For these purposes, will re-write the equation (37) in form of:

$$u_x(y,t) = \sum_{k=1}^{\infty} A_k(t)\cos(\pi\frac{(2k-1)}{2}y),\quad (0<y<1), \tag{42}$$

where

$$A_k(t) = \left[C_k + \frac{(-1)^{k+1}4F_k(t)}{\pi(2k-1)}\right]\cdot \exp\left(-\frac{\pi^2(2k-1)^2}{4\,Re}t\right). \tag{43}$$

Condition of the orthogonality of the function system $\cos(\pi\frac{(2k-1)}{2}y),\quad (k=1,2,...)$ will be:

$$\int_0^1 \cos(\pi\frac{(2k-1)}{2}y)\cdot\cos(\pi\frac{(2m-1)}{2}y)dy = \begin{cases} 0, & (k\neq m); \\ \frac{1}{2}, & (k=m). \end{cases}$$

consequently, the function system,

$$\psi_k(y) = \sqrt{2}\cos(\pi\frac{(2k-1)}{2}y),\quad (k=1,2,...)$$

at an interval [0;1] forms the normalized orthogonal system

$$\int_0^1 \psi_k(y)\cdot\psi_m(y)dy = \begin{cases} 0, & (k\neq m); \\ 1, & (k=m). \end{cases} \tag{44}$$

Considering the equality (44), will re-write the series (43) in form of:

$$u_x(y,t) = \sum_{k=1}^{\infty}\frac{A_k(t)}{\sqrt{2}}\cdot\psi_k(y),\quad (0<y<1), \tag{45}$$

according to which the Percival equality is obtained:

$$\int_0^1 u_x^2(y,t)dy = \frac{1}{2}\sum_{k=1}^{\infty}[A_k(t)]^2\ . \tag{46}$$

substituting the coefficient value $A_k(t)$ in the equation (46), will obtain:

$$\int_0^1 u_x^2(y,t)dy = \frac{1}{2}\sum_{k=1}^{\infty}\left[C_k + \frac{(-1)^{k+1}4F_k(t)}{\pi(2k-1)}\right]^2\cdot\exp\left(-\frac{2\pi^2(2k-1)^2}{4\,Re}t\right),\quad 0\le t<\infty. \tag{47}$$

Thus, for the quantity coefficient of flow will eventually obtain:

$$\beta = \frac{1}{2V^2}\sum_{k=1}^{\infty}\left[C_k + \frac{(-1)^{k+1}4F_k(t)}{\pi(2k-1)}\right]^2\cdot\exp\left(-\frac{2\pi^2(2k-1)^2}{4\,Re}t\right),\quad 0\le t<\infty, \tag{48}$$

where $\mathbf{V}$ – average instantaneous speed of the effective cross-section.

Coefficient of maldistribution of the instantaneous speeds

$$\alpha = \frac{4}{3\pi V^3}\left[\sum_{k=1}^{\infty}\left[C_k + \frac{(-1)^{k+1}4F_k(t)}{\pi(2k-1)}\right]\cdot\exp\left(-\frac{\pi^2(2k-1)^2}{4\,Re}t\right)\right]^3. \tag{49}$$

According to formulas (48) and (49), β=1, α=1 in t=0 $\beta\to$1,2, $\mathbf{a}\to\frac{54}{35}$ in $t\to\infty$, which corresponds to steady-state condition.

Conclusion

The results of the performed researches allow to conclude the following:

1. Regularity of the instantaneous speed distribution along the effective cross-section in the initial arbitrary distribution of speeds and in pressure differential allowing to calculate similar regularities in any practical problems relating to the automated system devices and pressure lines as well as lubricating systems for which basically the models of plane-parallel pressure flow is acceptable is obtained.
2. Regularity of the speed profile change (deformation) by time is obtained.
3. Regularity of average speed change is obtained.
4. Nature of change of quantity coefficients of flow β and kinetic energy α is specified.
5. Head losses complying to the nonstationary flow are determined.

REFERENCES

1. Громека И.С. К теории движения жидкости в узких цилиндрических трубах. М.:изд-во АН СССР. 1952.
2. Лойцянский Л.Г. Механика жидкости и газа. М.:Наука. 1973.
3. Бейтмен Г., Эрдейн А. Высшие трансцендентные функции. Изд. 2-е. М.:Наука. 1974.
4. Тихонов А.Н., Самарский А.А. Уравнения математической физики. М.: 1999.
5. Слезкин Н.А. Динамика вязкой несжимаемой жидкости. М.: 1955.
6. Тарг С.М. Основные задачи теории ламинарных течений. М.:Л.: 1951.

Arestak SARUKHANYAN, Doctor of Technical Science, Associate Professor,
Ani MANUKYAN, Chief Specialist
State University of Architecture and Construction of Erevan
E-mail: ani_manukyan@yahoo.com