ABOUT SOME ISSUES OF ANALYSIS OF EQUATIONS OF DYNAMICS OF COMPOSITE WATER-SOIL MIXTURE

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Equation system of water-soil mixture motion proposed by Professor T. G. Voynich-Sianozhenski is reviewed. Some solutions of the equation system of linear and nonlinear statements elaborated by the author are provided. Methods of reducing them to Burgers equation system and then to general positions of heat conductivity are presented.

Calculations for estimating wave parameters according to various methods are performed. The first method includes numeric solution of the equation system of the motion according to the method of characteristics earlier worked out by the author, the second – approximate solution of the system according to Dressler model, the third is based on application of perturbation method and finally, the forth uses both reduction of basic equations to Burgers equation and then, as a consequence, to the equation of heat conductivity and the calculations of ready relations provided in the book by James Aussem "linear and nonlinear waves" (1977).

Key words: water-soil media, flow velocity, tractional load, system stability, bottom waves, composite media.

Equations of water-soil composite media (mixture of water and soil) motion are proposed by T. G. Voynich-Sianozhenski [1] and have the following form:

$$\frac{\partial}{\partial t}(\rho Q)+\frac{\partial}{\partial x}(\rho WQ)=2\mu(1-s)\omega\frac{\partial^2 W}{\partial x^2}+g\rho(i_0-i_f)\omega+BT-B(\tau_0+ks\cos^2\varphi)-g\rho\omega\frac{\partial h}{\partial x}\cos\psi-$$

$$-\frac{\omega}{1+2f^2}\frac{\partial P}{\partial x}+2g(\rho_s-\rho_w)\frac{s\omega\sin^2\varphi}{1+\sin^2\varphi}\frac{\partial h}{\partial x}\cos\psi-g(\rho_s-\rho_w)\frac{s\omega\sin\varphi\cos\varphi}{1+\sin^2\varphi}\frac{\partial h}{\partial x}\cos\psi;$$

$$\frac{\partial\omega}{\partial t}+\frac{\partial Q}{\partial x}=0\ , \qquad (1)$$

where Q - discharge; W - average velocity; $h,i,\psi,B,\omega-$ geometric characteristics of mixture flow; $s,\varphi,\mu,k,f,\rho-$ physical parameters of water-soil media [2].

First of all, such a model is aimed at describing behavior of the bottom drift, however it is also suitable for estimating the motion of viscous fluid and the behavior of sand and mixtures of viscous fluid and sand in various concentrations.

Among practical problems that may be reviewed based on a model equation is the problem on the motion of the tractional load layer, the issues of describing the behavior of soil that is close to bridge supports, erosions and deformations at the river bank and sea coast zones, etc.

In other words the model is applied when from physical or other viewpoint, the stylization of a specific problem relating to the description of the water-soil mixture behavior within the one-dimensional equations is admitted.

Large range of the problems that were reviewed and may be reviewed in future, as we believe, demonstrates the interest to study particularly the equations of model.

In the first section of the work, basic results – solution of the equation system (1) for simpler cases (in terms of practical application) mainly obtained based on simplified linearized equation are provided.

The second section of the works reviews nonlinear equation systems (1).

It is shown that the nonlinear equation system of the model is similar to Burgers equation with variable coefficients, i.e. can be considered as modified Burgers equation. A model of approximate resolution of the system of the respective equations is drafted here as well.

Some practical results of the analysis of the nonlinear equations for some cases are provided in [2,3].

First of all, for stationary longitudinal uniform motion of the water-soil mixture layer (tractional load) with the thickness h, from (1) will obtain

$$g\rho i_0\omega + BT - B(\tau_0 + ks\cos^2\varphi) - g\rho\omega i_s - g(\rho_s - \rho_w)\frac{sf\omega}{1+2f^2}\cos\psi = 0 . \qquad (2)$$

Using Chezy formula for tangential stresses at water and soil boundary accepted in hydraulics

$$T = \frac{g}{C_H^2}\rho_w (V - W)^2 ,$$

where W - velocity of the water-soil layer; V – velocity of water flow, will obtain minimum value of the flow velocity under which mass motion of drifts (layer thickness with one average particle diameter $h = d$) is initiated.

For researching the conditions of the stability of stationary longitudinal uniform motion of the layer of tractional load with thickness h in relation with small perturbations, the following linear equations was obtained

$$\frac{\partial^2\varsigma}{\partial t^2} + P_1\frac{\partial^2\varsigma}{\partial t\partial x} + P_2\frac{\partial^2\varsigma}{\partial x^2} + P_3\frac{\partial\varsigma}{\partial t} + P_4\frac{\partial\varsigma}{\partial x} = 0 \qquad (4)$$

where,

$$P_1 = 2\alpha_0 W_0;$$

$$P_2 = \alpha_0 W_0{}^2 - \frac{gh_0}{1+2f^2}\frac{(\rho - \rho_w(1+2f^2))}{\rho}\cos\psi;$$

$$P_3 = \frac{2\rho_w g(V_0 - W_0)}{\rho h_0 C_H^2};$$

$$P_4 = \frac{1}{\rho h_0 C_H^2}\left[\begin{array}{l} C_H^2(\tau_0 + ks\cos^2\varphi) + 2\rho_w g W_0(V_0 - W_0) + \rho_w(V_0 - W_0)^2\frac{gh_0}{3H_0} + \\ + \frac{3\rho_w g V_0(V_0 - W_0)}{H_0} - \rho_w g(V_0 - W_0)^2 \end{array}\right]$$

Index 0 carries respective value to the stationary longitudinal uniform motion.

Solutions of the equation (4) are stable with respect to small perturbations when the average flow velocity $\mathbf{V}$ satisfies an inequality $\mathbf{V > V_2}$, where

$$\mathbf{V_2 = 1.5V_1 - \frac{z}{2g\rho_w V_1} + \sqrt{\frac{\alpha_0 (g\rho_w V_1^2 - z)^2 + 4bV_1^2 g^2 \rho_w{}^2}{4(\alpha_0 - 1) g^2 \rho_w{}^2 V_1^2}};}$$
$$\mathbf{z = C_H^2 (\tau_0 + ks\cos^2\varphi), b = \frac{(\rho_s - \rho_w)}{\rho_w}\frac{sfh}{1+2f^2}\cos\psi.} \qquad (5)$$

An essence of the obtained result is that in water flow velocities $\mathbf{V > V_1}$, that are more than the first critical but less than the second critical $\mathbf{V < V_2}$, the stationary longitudinal uniform motion is unstable, i.e. the wave - ridge motion of the tractional load occurs. In $\mathbf{V > V_2}$ the ridge drawdown i.e. return to non-wave carpet motion of drifts occurs.

Solution of the nonlinear equation system (1) for the flow velocities within the range of $\mathbf{V_2 > V > V_1}$,, i.e. determination of the parameters of the wave motion of the bottom layer were performed in several ways.

Numerical solution of the equation system (1) by means of the method of characteristics is described in detail in [2].

Approximate solution of the system (1) based on Dressler – Mkhitaryan diagram allowed to estimate geometric parameters (maximum height, length, steepness) for quasisteady motion of the bottom waves.

It was obtained that the wave height $\mathbf{h}$ is such that $\mathbf{h_1 < h < h_2}$.

where,

$$\mathbf{h_1 = \frac{z_0}{2} h^* (2 + z_0 - \sqrt{z_0{}^2 + 4z_0}),}$$
$$\mathbf{h_2 = \frac{z_0}{2} h^* (2 + z_0 + \sqrt{z_0{}^2 + 4z_0});}$$
$$\mathbf{z_0 = \sqrt{\frac{gG\rho_w}{C_H{}^2 M}};}$$
$$\mathbf{h^* = \frac{(V_0 - c)^3}{Gz^2};}$$
$$\mathbf{M = \frac{sfg(\rho_s - \rho_w)}{1+2f^2}\cos\psi;}$$
$$\mathbf{G = \frac{g}{\rho}\left[\rho - \frac{\rho_w}{1+2f^2} - \frac{2sf^2(\rho_s - \rho_w)}{1+2f^2}\right]\cos\psi;}$$

c – velocity of quasisteady motion of the bottom waves; h_1, h_2 - respective minimum and maximum height of the bottom layers.

Let's show basic segments of the approximate solution of the system (1) which is based on application of small parameter method.

Self-similar transformation $\xi = x - ct$ converts the equation system (1) into the following system of ordinary differential equations:

$$\frac{d}{d\xi}\big[(W-c)W\big] = 2\nu h\frac{d^2W}{d\xi^2} + gh(i_0 - i_f) + \frac{T-(\tau_0 + ks\cos^2\varphi}{\rho} + h\frac{\rho_w}{\rho}\frac{g}{1+2f^2}\frac{dh}{d\xi} + \frac{(\rho_s-\rho_w)}{\rho_w}\frac{2ghf^2}{1+2f^2}\cos\psi - gh\frac{dh}{d\xi}\cos\psi - \frac{(\rho_s-\rho_w)}{\rho_w}\frac{sghf}{1+2f^2}\cos\psi \qquad (6)$$

$$(W-c)h = q_0 = const\,. \qquad (7)$$

Here, ν - coefficient of effective dynamic viscosity; $\nu = \frac{\mu}{\rho}(1-s)$.

By substituting **W** from (7) by (6) and conducting simplification, will obtain

$$2\nu q_0\big[(hh'' - 2(h')^2\ \big] = q_0{}^2\frac{dh}{d\xi} + A_1h^3\frac{dh}{d\xi} - A_2h^3 + \frac{\rho_w g\big[(V-c)h - q_0\ \big]^2}{\rho C_H^2} - (\tau_0 + ks\cos^2\varphi)h^2\ ; \qquad (8)$$

where,

$$A_1 = \frac{2(\rho_s-\rho_w)}{\rho_w}\frac{gf^2}{1+2f^2}\cos\psi - g\cos\psi + \frac{\rho_w}{\rho}\frac{g}{1+2f^2}$$

$$A_2 = \frac{(\rho_s-\rho_w)}{\rho_w}\frac{gf}{1+2f^2}\cos\psi + g(i_0 - i_f).$$

Solution of the linear equation (8) was obtained by small parameter method according to ν degrees in form of

$$h(\xi) = \sum_{i=1}^{\infty} h_i(\xi)\nu^i\ . \qquad (9)$$

Equating the coefficients in equal ν degrees from the equation (8) will obtain the following relations (let’s restrict with the first two members of order (9)):

$$q_0{}^2\frac{dh_0}{d\xi} + A_1h_0^3\frac{dh_0}{d\xi} - A_2h_0^3 + A_3\big[(V-c)h_0 - q_0\ \big]^2 - (\tau_0 + ks\cos^2\varphi)h_0^2 = 0\,. \qquad (10)$$

$$q_0\big[(h_0h_0'' - 2(h_0')^2\ \big] = = q_0{}^2\frac{dh_1}{d\xi} + A_1h_1{}^3\frac{dh_1}{d\xi} - A_2h_1{}^3 + A_3\big[(V-c)h_1 - q_0\ \big] - (\tau_0 + ks\cos^2\varphi)h_0^2; \qquad (11)$$

$$A_3 = \frac{g\rho_w}{\rho C_H^2}\,.$$

Relation (10) can be re-written in a following way:

$$\xi - \xi_0 = \int\frac{(q_0^2 + A_1h_0^3)dh_0}{A_2h_0^3 - A_3\big[(V-c)h_0 - q_0\ \big]^2 + (\tau_0 + ks\cos^2\varphi)h_0^2}\ . \qquad (12)$$

Quasisteady bottom waves which we are interested in are feasible just in case in the equation

$$A_2h_0^3 - A_3\big[(V-c)h_0 - q_0\ \big]^2 + (\tau_0 + ks\cos^2\varphi)h = 0 \qquad (13)$$

one root is real $h = h_1^*$, and the other two are complex conjugate $h = \alpha \pm \beta i$.

Integral of the equation (12) can be written in a following way:

$$\xi - \xi_0 = \frac{A\ h_0}{A_2} + B_1 \ln\left|\frac{h}{h_1^*} - 1\right| + B_2 \ln\left|1 - \frac{2\alpha}{h_1^*} + \frac{\alpha^2 + \beta^2}{h_1^{*2}}\right| + C_1 \mathrm{arctg}\frac{h_0 - \alpha}{\beta}, \qquad (14)$$

from where the relevant expressions for h_0 and h_1 are found.

Let's adduce the relation for h_0:

$$h_0 = \alpha + \beta \mathrm{tg}\{ \frac{A_1 h_0}{A_2 C_1} + \frac{B_1}{C_1}\ln\left|\frac{h}{h_1^*} - 1\right| + \frac{B_2}{C_1}\ln\left|1 - \frac{2\alpha}{h_1^*} + \frac{\alpha^2 + \beta^2}{h_1^{*2}}\right| - \frac{\xi_0 - \xi}{C_1}\} \ . \qquad (15)$$

Solutions obtained based on (15) quite well comply with the available data of the experiments with caproic chips [3].

In the following section it is determined that the equation system (1) is reduced to one of the most famous type of nonlinear equations – to Burgers equation with the variable coefficients.

For this, let's write the equation system (1) in form of:

$$\frac{\partial h}{\partial t} + w\frac{\partial h}{\partial x} + h\frac{\partial w}{\partial x} = 0;$$

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = 2\nu h\frac{\partial^2 w}{\partial x^2} + g(i_0 - i_f)h + \frac{T - B(\tau_0 + ks\cos^2\varphi)}{\rho} + A_1 h\frac{\partial h}{\partial x} + A_2 h \ . \qquad (16)$$

Then, according to the model of transformed nonlinear equations developed in [4], let's re-write the equation system (16) in a matrix form

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial t} + Z\frac{\partial^2 U}{\partial t^2} + BS_x = 0. \qquad (17)$$

Here, U and S – vectors, A, B and Z – matrixes; $U=\begin{pmatrix} h \\ w \end{pmatrix}$ - vector of unknowns;

$$A=\begin{pmatrix} w & h \\ A_1 & w \end{pmatrix};\ B=\begin{pmatrix} 0 \\ \dfrac{\tau_0 + ks\cos^2\varphi - T}{\rho h} \end{pmatrix};\ \ Z=\begin{pmatrix} 0 & 0 \\ 0 & -2\nu(1-s) \end{pmatrix}.$$

Presenting vector U in form of

$$U = U_0 + U_1 \exp(ikx - i\sigma t), \qquad (18)$$

will place (18) in (17). Will obtain

$$\left\{-(\frac{\sigma}{k})I + A_0 ikZ\right\}U_1 = 0. \qquad (19)$$

Here and in future, index 0 carries relevant value to the stationary motion.

Equation (19) is solved by the method of consequent approximations.

In zero approximation, will have

$$\left\{-(\frac{\sigma}{k})I + A_0\right\}U_{10} = 0; \qquad (20)$$

U_{10} - one of right latent matrix vectors A_0 corresponding to eigenvalue $c_0 = \frac{\sigma}{k}$.

We have $\mathbf{c_0 = \frac{\sigma}{k} = w_0 + \sqrt{A_1 h_0}}$;

$\mathbf{l_0 = (\sqrt{A_1}, \sqrt{h_0}), r_0 = (\sqrt{h_0}, \sqrt{A_1})}$ – left and right latent matrix vectors $\mathbf{A_0}$.
By placing the obtained expressions in (19) and multiplying by l_0 from left, will obtain

$$\mathbf{c_1 = \frac{\sigma}{k} = c_0 + \frac{irl_0 Z_0 r_0}{l_0 r_0}} . \quad (21)$$

Further, according to [4], will proceed from (x,t) to (ς, η) by means of the relation

$$\varsigma = \varepsilon\left(\int \frac{\mathbf{dx}}{\mathbf{c_0}} - \mathbf{t}\right),\ \eta = \varepsilon^2 \mathbf{x} .$$

Assuming that k is small (approximation of long waves) and using the method of consequent approximations according to the degrees of parameter $\varepsilon \approx \mathbf{k}$ of small parameter characterizing the degree of system nonlinearity

$$\mathbf{A = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + ...;}$$
$$\mathbf{B = A_0 + \varepsilon B_1 + \varepsilon^2 B_2 + ...;}$$
$$\mathbf{K = K_0 + \varepsilon K_1 + \varepsilon^2 K_2 + ...,}$$

will adduce the system (16) to Burgers equation with the variable coefficients

$$\mathbf{u_\eta + \alpha \exp(-\chi\eta) u u_\zeta = \beta u_{\zeta\zeta}} , \quad (22)$$

where,

$$\alpha = \frac{\mathbf{3}}{\mathbf{2c_0^2}}, \beta = \frac{\mathbf{(1-s)\nu}}{\mathbf{c_0^2}} .$$

Procedure of the reduction of the system (16) to the equation (22) is in detail described in [5]. The symbols used in (22) are also described there in detail.

By means of replacing $\mathbf{u = \varphi(\varsigma, \eta) \exp(\chi\eta)}$ the equation (3) is adduced to modified Burgers equation with constant coefficients.

$$\varphi_\eta + \chi\varphi + \alpha\varphi\varphi_\zeta = \beta\varphi_{\zeta\zeta} . \quad (23)$$

It should be noted, that constant coefficient values in the equations (22), (23) are taken based on respective parameters in longitudinal uniform motion.

For two important private cases 1) when coefficient χ is neglible and 2) high profiles of the variable η are considered (asymptotic behavior of the solution when $\eta \to \infty$), the equations (22), (23) are reduced to the heat conductivity equation.

For this, by means of replacing Cole-Hopf $\varphi = -2\beta \frac{\Psi_\zeta}{\Psi}$, the equation (22) is transformed into the equation

$$\psi_\eta = \mathbf{a}^2 \psi_{\varsigma\varsigma} , \quad (24)$$

I.e. into standard heat conductivity equation.

Note, that for the problem involved the reduction of initial equation system (1) to Burgers equation and then to the linear equation – the equation of heat conductivity – allows to carry the existing solutions of this equation.

The most important for such cases is description of the stationary waves, N – waves, single hump type waves and discontinuous solutions. At the same time, in accordance with [6], the solutions for fluids with various Reynolds numbers were reviewed.

Calculations for estimating the wave parameters with various methods were carried out.

First method – numeric solution of the equation system (1) with the method of characteristics provided in [2], second model is based on approximate solutions of the system (1) with Dressler's model, i.e. based on the relations for identifying the bottom wave parameters again provided in detail in [2].

Solution of the system (1) on the basis of the small parameter method was an essence for the third method. And, finally, the forth method used as a basis the reduction of basic equations of the problem in relevant cases to Burgers equation and respectively to heat conductivity equation and calculations with ready relations [6].

Comparison of the obtained calculation results for quasistationary waves (lengths, heights and profiles of the bottom waves for two model cases) showed, that accepted coincidence for the first two models the difference in non-dimensional parameter $\mathbf{z} = \mathbf{h}/\lambda$ did not exceed 35%.

The third and forth models gave a bit larger discrepancy.

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