INVESTIGATION ON NON-STATIONARY FLOW IN CYLINDRICAL CHANNELS OF CIRCULAR CROSS-SECTION

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The primary problem of unsteady flows' study is calculation of hydraulic losses. A formula for calculation of hydraulic losses in unsteady pressure motion has been presented in the present work. To evaluate each term of the formula and perform calculation of energy loss it is necessary to study structural changes running in unsteady flows. The present paper presents the results of investigation of unsteady laminar flow of viscous fluid in a cylindrical pipe of circular cross-section. Integrating the differential equation of axial-symmetric pressure flow of viscous fluid regularities of hydraulic parameters change in general boundary and initial conditions have been obtained. On the basis of general solutions fomulae for calculation of hydraudynamic parameters of unsteady pressure flow when the pressure gradient is changed instanteneously. Computer-aided experimental study graphs were plotted for instanteneous change of velocity, shearing stresses, average velocity of the flow, coefficients of momentum and kinetic energy. According to the results of calculations conclusions have been drawn on the nature of structural changes and formation of hydraulic losses.

Losses of energy occurring in cylindrical channels of circular cross-section in case of non-stationary flow is calculated by the following formula

$$hw = -\frac{1}{\rho g}\frac{\partial p}{\partial x}\cdot l + \frac{l}{g}\left(\beta\frac{\partial V}{\partial t} + \frac{V}{2}\frac{\partial \beta}{\partial t}\right), \tag{1}$$

where β is the factor of kinetic momentum change

$$\beta = \frac{\int_{A_0} u^2\, dA}{V^2 A_0}: \tag{2}$$

Eq. (1) can be represented as

$$hw = hw_0 + h_{i_1} + h_{i_2}, \tag{3}$$

where hw_0 is the loss of energy originated in uniform flow to which tends the given non-uniform flow, $h_{i1} = \beta\frac{\partial V}{\partial t}\frac{l}{g}$ is the inertia head conditioned by velocity change, $h_{i2} = \frac{lV}{2\cdot g}\cdot\frac{\partial \beta}{\partial t}$ is the inertia head conditioned by velocity diagram deformation[3].

Thus, the meaning of energy loss is not the same in cases of non-uniform non-stationary and stationary flow. During non-stationary flow a part of the specific energy of the fluid is spent on overcoming friction forces, and transforming to heat energy is absorbed by ambient medium and is not restored, as for the other part - it is used on overcoming the fluid inertia and velocity diagram deformation, which is restored with time.

In case of non-stationary flow all parameters of the flow, therefore, including energy loss depend on time and can be determined. If it is assumed that non-stationary flow is continuation of quasi-stationary flow which suppose parabolic distribution of velocities [3], then we have

$$\beta = 1{,}33, \quad \frac{\partial \beta}{\partial t} = 0, \quad hw_h = \frac{2\tau_0 l}{\rho g r_0}.$$

Eq. (1) now can be expresses as

$$\frac{2\tau_0 l}{\rho g r_0} = -\frac{1}{\rho g}\frac{\partial p}{\partial x}\cdot l + \frac{1{,}33\cdot l}{g}\frac{dv}{dt} \tag{4}$$

In the result of non-stationary flow study on the basis of quasi-stationary model a formula was derived in the form of transfer function for calculation of shear stresses occurring on the channel fixed wall [3]. However, as it was mentioned above the quasi-stationary flow model is not correctly characterized the real behaviour of flow and energy losses initiation mechanism [1].

In non-uniform non stationary flow the occurring energy loss should be calculated by Eq.(1) for which it is necessary to know regularity of velocity non-uniform distribution in the flow section which enables to obtain the variation function of energy losses.

Often in pressure systems due to pressure gradient $\frac{\partial P}{\partial x}$ change the flow stationary regime is violated causing time-dependent change in hydromechanical parameters. In consequence of pressure gradient change under influence of friction and inertia varying forces non-stationary flow of fluid occurs. Research of these phenomena are of great practical and theoretical interest.

Investigation of non-stationary laminar flow in a cylindrical pipe of circular cross-section is reduced to integration of Navier-Stokes equations which in case of axially-symmetrical flow in the cylindrical coordinate system is given by [2,5,6]

$$\frac{\partial U}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right), \quad \frac{\partial P}{\partial \theta} = 0, \quad \frac{\partial P}{\partial r} = 0, \tag{5}$$

where U is a velocity component along the axis of the cylinder and all other components are equal to zero. For an incompressible fluid from the continuity equation of it follows

$$\frac{\partial U}{\partial x} = 0,$$

which means that the velocity component along the axis of the cylinder is not dependent on x coordinate.

From the last two equation it follows that the pressure function depends only on x coordinate and t time. Therefore for any fixed section $(x = const)$ pressure change will be dependent on time only, hence

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = f(t) \tag{6}$$

To integrate Eq. (5) the initial and boundary conditions also are given. As an initial condition the function of velocity distribution in the flow section of the pipe is given at the starting moment. Let us now consider the general case when in $x = 0$ section of the pipe velocities are distributed according to an arbitrary function, that is $U|_{x=0} = \varphi(r)$. It is assumed that viscous fluid adheres to the wall of the pipe, hence, it is immobile and consequently $U(R,t) = 0$, where R is the radius of the pipe.

Introducing dimensionless values and coordinates

$$U = U_\infty U_0; \qquad x = Rx_0;$$
$$r = Rr_0;$$
$$t = \frac{R}{U_\infty} t_0; \qquad P = \rho U_\infty^2 P_0;$$

where $U_\infty = \lim_{t\to\infty} U(0,t)$, Eq. (5) now is given by

$$\frac{\partial U}{\partial t} = f(t) + \frac{1}{Re}\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right), \tag{7}$$

where $Re = \frac{U_\infty \cdot R}{\nu}$ is the Reynolds number.

The obtained initial and boundary conditions of the differential equation are, respectively

$$\begin{aligned} &U(r,t)\big|_{t=0} = \varphi(r), \qquad (0 \le r \le 1; \quad \varphi(1) = 0) \\ &U(r,t)\big|_{t=1} = 0 : \end{aligned} \tag{8}$$

The solution of Eq.(7) is sought for

$$U(r,t) = \sum_{k=1}^{\infty} C_k(t) J_0(rq_k) \tag{9}$$

in the form of an infinite series where $C_k(t)$ unknown factors are functions which take into account velocity deviations from velocities of stationary flow, $J_0(rq_k)$ is the Bessel first kind zero order function, q_k are roots of $J_0(rq_k)=0$ equation.

The general solution of the problem is obtained as

$$U(r,t) = \sum_{k=1}^{\infty}\left(C_k + \frac{2}{q_k J_1(q_k)}\int_0^t \exp\left(\frac{q_k^2}{Re} t\right) f(t) dt\right) \cdot J_0(rq_k) \exp\left(-\frac{q_k^2}{Re} t\right) = \sum_{k=1}^{\infty} A_k(t) J_0(rq_k), \tag{10}$$

where

$$A_k(t) = \left(C_k + \frac{2}{q_k J_1(q_k)}\int_0^t \exp\left(\frac{q_k^2}{Re} t\right) f(t) dt\right) \exp\left(-\frac{q_k^2}{Re} t\right) \tag{11}$$

Accelerating friction stresses between viscous fluid layers are determined by Newton's law

$$\tau = \pm\mu \frac{dU}{dr} : \tag{12}$$

Therefore the regularity of sear stresses change in case of non-stationary laminar flow is expresses by

$$\tau = \mu \sum_{k=1}^{\infty} A_k(t) q_k J_1(rq_k) : \tag{13}$$

C_k factors are determined employing the initial (8) condition of the problem

$$C_k = \frac{2}{J_1^2(q_k)} \cdot \int_0^1 \varphi(r) \cdot r J_0(rq_k) dr : \tag{14}$$

General solutions (10) and (14) of the problem enable obtaining solutions for particular cases. Having the regularity of velocity distribution the average velocity in the flow section can be found

$$V = 2\int_0^1 rU(r,t)dr = 2\sum_{k=1}^{\infty} A_k(t)\cdot\frac{J_1(q_k)}{q_k}: \tag{15}$$

To determine the momentum variation coefficient using Eqs.(10) and (15) we have

$$\beta = 2\int_0^1 \frac{rU^2 dr}{V^2}: \tag{16}$$

Employing Parseval's equation [4] for orthogonal series the value of the last integral is determined

$$\beta = \frac{\sum_{k=1}^{\infty} A_k^2(t)[J_1(q_k)]^2}{4\left[\sum_{k=1}^{\infty}\frac{J_1(q_k)}{q_k}A_k(t)\right]^2}: \tag{17}$$

In case of pressure variation constant $-\frac{1}{\rho}\frac{\partial P}{\partial x} = const$**; for velocity distribution regularity we get**

$$\begin{aligned} U(r,t) &= \sum_{k=1}^{\infty}\left(C_k - \frac{2\mathrm{Re}\frac{\partial P}{\partial x}}{q_k^3 J_1(q_k)}\left(\exp\left(-\frac{q_k^2}{\mathrm{Re}}t\right)-1\right)\right)\cdot J_0(rq_k)\cdot\exp\left(-\frac{q_k^2}{\mathrm{Re}}t\right) = \\ &= \sum_{k=1}^{\infty}\left(\left(C_k + \frac{2\mathrm{Re}\frac{\partial P}{\partial x}}{q_k^3 J_1(q_k)}\right)\exp\left(-\frac{q_k^2}{\mathrm{Re}}t\right) - \frac{2\mathrm{Re}\frac{\partial P}{\partial x}}{q_k^3 J_1(q_k)}\right)\cdot J_0(rq_k) \end{aligned} \tag{18}$$

when $t \to \infty$ **the non-stationary flow becomes stationary. From Eq. (18) the regularity of velocity distribution in case of stationary flow is obtained. Making** $t \to \infty$ **boundary passage we obtain**

$$U_{st}(r) = -2\mathrm{Re}\frac{\partial P}{\partial x}\sum_{k=1}^{\infty}\frac{J_0(rq_k)}{q_k^3 J_1(q_k)}, \tag{19}$$

or

$$U_{st}(r) = -\frac{\mathrm{Re}}{4}\frac{\partial P}{\partial x}(1-r^2) = U_{\infty}(1-r^2): \tag{20}$$

Here $U_{\infty} = -\frac{\mathrm{Re}}{4}\frac{\partial P}{\partial x}$ **is the maximum velocity in case of stationary flow, that is velocity in the centre of the pipe.**

Thus, regardless of the initial pattern of velocity distribution in a cylindrical pipe of circular cross-section, velocity distribution regularity of originating non-stationary flow velocity distribution in case of pressure variation constant value when $t \to \infty$ **tends to the known second-order parabola law.**

Let us now examine accelerating flow of real fluid in a pipe of constant diameter, when the fluid and the walls of the pipe are not deformable and at terminal points on l length of the pipe pressure constant values $(P_1$ and $P_2)$ are present. At the initial moment the fluid is in the state of rest - $\varphi(r)=0,$, therefore $C_k(t)=0$ and $f(t)=-\frac{\partial P}{\partial x}=\frac{R}{U_\infty^2}\frac{P_1-P_2}{\rho l}=P=\text{const}.$

Substituting values of $C_k(t)$ and $f(t)$ functions into Eq.(11) we get

$$A_k(t)=\frac{2\,\text{Re}\,P}{q_k^3 J_1(q_k)}\left(1-\exp\left(-\frac{q_k^2}{\text{Re}}t\right)\right) \tag{21}$$

Having values of $A_k(t)$ coefficients we can determine the regularity of velocity variation

$$U(r,t)=\sum_{k=1}^{\infty}\frac{2\,\text{Re}\,P}{q_k^3 J_1(q_k)}\left(1-\exp\left(-\frac{q_k^2}{\text{Re}}t\right)\right)J_0(q_k r): \tag{22}$$

The average velocity of the section will be

$$V(t)=4\sum_{k=1}^{\infty}\frac{\text{Re}\,P}{q_k^4}\left(1-\exp\left(-\frac{q_k^2}{\text{Re}}t\right)\right): \tag{23}$$

From Eqs.(12) and (22) the law of shear stresses variation is obtained

$$\tau(r,t)=\sum_{k=1}^{\infty}\frac{2\mu\,\text{Re}\,P}{q_k^2 J_1(q_k)}\left(1-\exp\left(-\frac{q_k^2}{\text{Re}}t\right)\right)J_1(q_{kt}): \tag{24}$$

For the momentum variation factor according to Eq.(17) we have

$$\beta=\frac{\sum_{k=1}^{\infty}A_k^2(t)[J_1(q_k)]^2}{4\left[\sum_{k=1}^{\infty}A_k(t)\frac{J_1(q_k)}{q_k}\right]^2}=\frac{\sum_{k=1}^{\infty}\frac{1}{q_k^6}\left(1-\exp\left(-\frac{q_k^2}{\text{Re}}t\right)\right)^2}{4\left(\sum_{k=1}^{\infty}\frac{1}{q_k^4}\left(1-\exp\left(-\frac{q_k^2}{\text{Re}}t\right)\right)\right)^2}. \tag{25}$$

The coefficient of velocities non-uniform distribution is calculated by

$$\alpha=\frac{1}{2}\frac{\int_0^1 r\left(\sum_{k=1}^{\infty}\frac{1}{q_k^3 J_1(q_k)}\left(1-\exp\left(-\frac{q_k^2}{\text{Re}}t\right)\right)\cdot J_0(q_k r)\right)^3 dr}{\left(\sum_{k=1}^{\infty}\frac{1}{q_k^4}\left(1-\exp\left(-\frac{q_k^2}{\text{Re}}t\right)\right)\right)^3}. \tag{26}$$

From the above equation it follows that when $t\to 0$, $\beta\to 1$, $\alpha\to 1$ and $t\to\infty$, $\beta\to 4/3$, $\alpha\to 2$, which corresponds to the stationary regime.

Values of $U(r,t)$, $V(t)$, $\beta(t)$, $\alpha(t)$ functions for different values of Reynolds number have been computed and plotted. Below are graphs for $\text{Re}=1, 100, 1500, 2000$ Reynolds number and different time values.

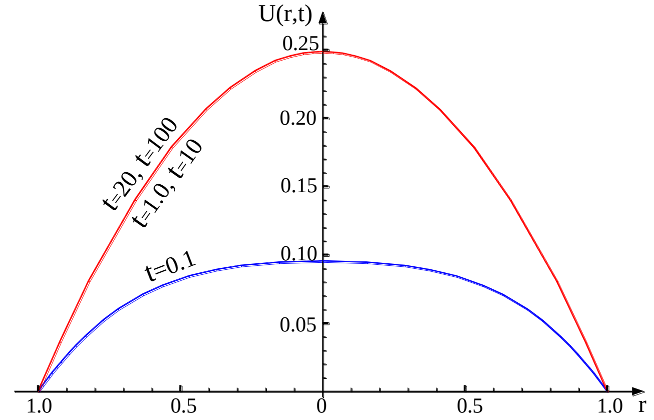

Fig. 1. Time-dependent velocity change when $\mathbf{Re = 1}$, $\mathbf{(t_{st} = 0.824)}$

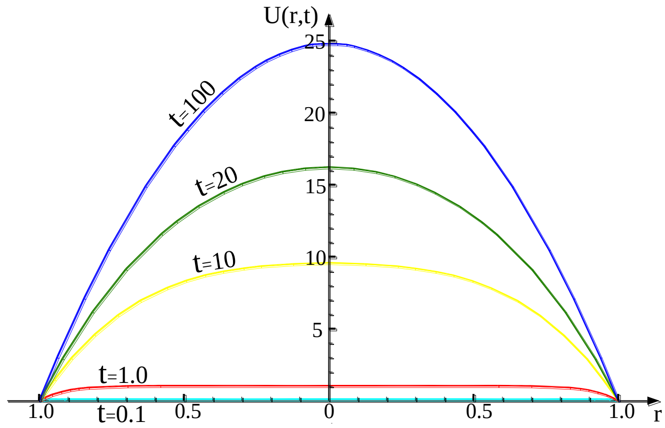

Fig. 2. Time-dependent velocity change when $\mathbf{Re = 100}$, $\mathbf{(t_{st} = 81.407)}$

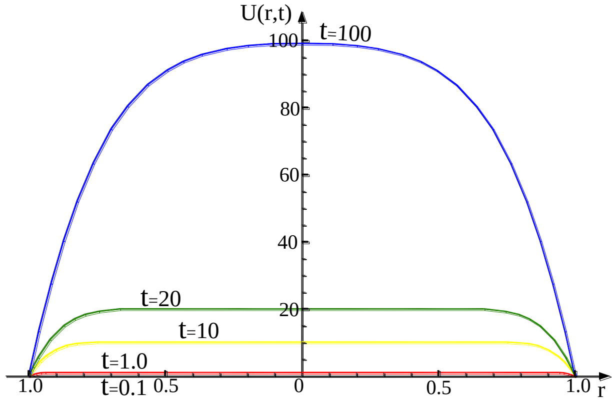

Fig. 3. Time-dependent velocity change when $\mathbf{Re = 1500}$, $\mathbf{(t_{st} = 1221.1)}$

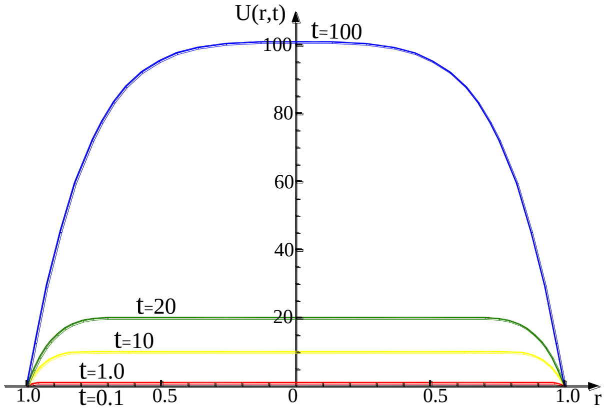

Fig. 4. Time-dependent velocity change when $Re = 2000,\ (t_{st} = 1628.13)$

It follows from the above graphs that for small Reynolds number non-stationary flow quickly tends the stationary flow regime. To each graph in case of stationary flow maximum values of velocity in the centre of the pipe are attached. Time values also are given, during to which in case of stationary flow the velocity in the pipe centre equals 0.99 part of maximum velocity.

From the graph representing velocity change it is not difficult to arrive at a conclusion that the fluid in state of rest begins moving in close to pipe walls layers. A boundary layer is developed gradually spreading its influence to the pipe centre. Due to viscous forces the velocity field gradually is spread through the entire section of the pipe.

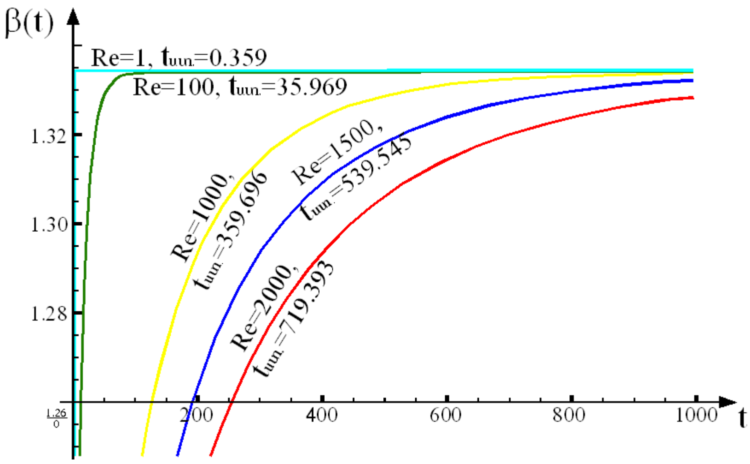

Fig. 5. $\beta(t)$ coefficient variation when $Re = 1, 100, 1000, 1500, 2000$

Fig. 5 represents momentum coefficient (β) variation graphs for Reynolds number different values. Variation boundaries of graphs are $1 \le \beta \le \frac{4}{3}$. In case of small values of Reynolds number β coefficient quickly tends to the value of stationary regime - 4/3. So much is the Reynolds number as much is delay of β coefficient tendency to its stationary regime value. Attached to the represented graphs for each value of Reynolds number time durations are calculated in which β coefficient becomes equal to $0.99\beta_{st}$.

Velocity non-uniform distribution coefficient α is calculated by

$$\alpha = \frac{2\int_0^R r\left(U(r,t)\right)^3 dr}{R^2\left(V(t)\right)^3}.$$

Fig. 6 represents computed values of α coefficient for various Reynolds numbers. To each graph time durations are given in which occurs stabilisation.

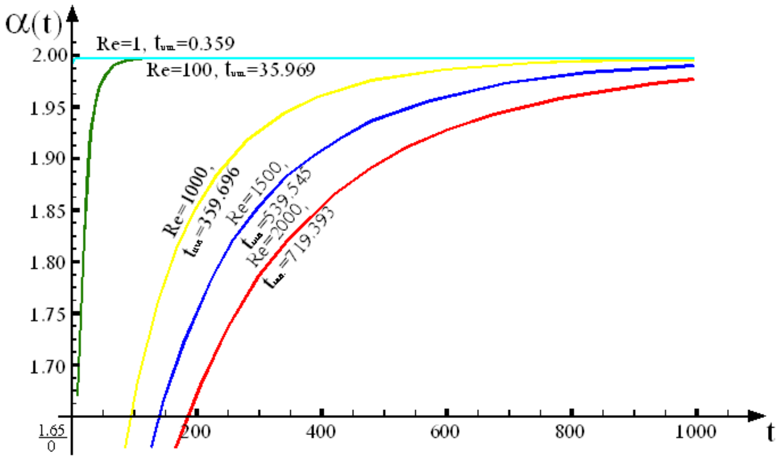

Fig. 6. Variation of $\alpha(t)$ coefficient when $Re = 1, 100, 1000, 1500, 2000$

Fig. 6 shows variation of the average velocity for four values of the Reynolds number ($Re = 1, 100, 1500, 2000$). For small values of the Reynolds number non-stationary flow quickly tends to stationary regime of the flow (curve 1). For each curve velocities and duration in which the velocity in the centre of the pipe becomes equal to 0,99 part of the maximum velocity have been calculated.

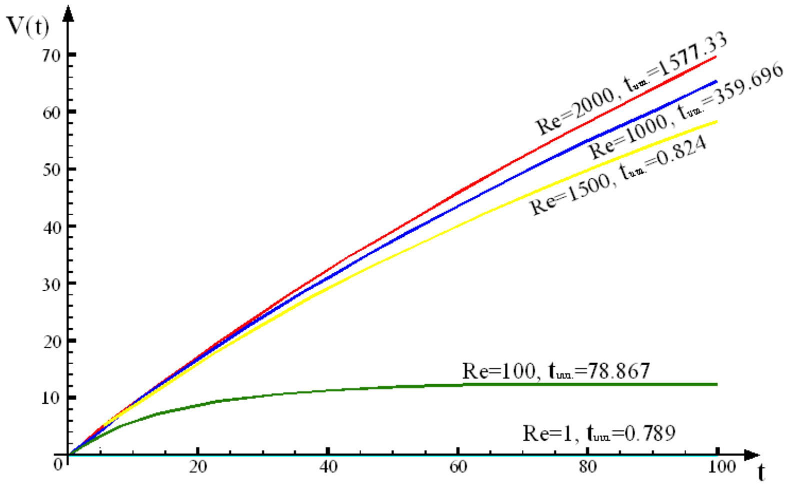

Fig. 7. Variation of the average velocity of the section when $Re = 1, 100, 1000, 1500, 2000$

Having of the average velocity and momentum coefficient variation regularities graphs of energy loss in case of non-stationary flow for the Reynolds numbers $Re = 1, 100, 1500, 2000$ have been plotted (Fig. 8).

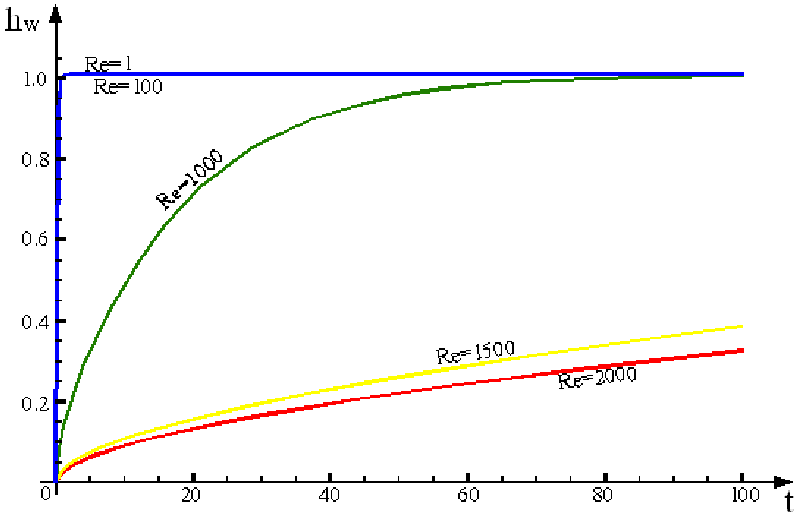

Fig. 8. Variation of energy loss when $Re = 1, 100, 1000, 1500, 2000$

The plotted graphs show that in non-stationary flow energy loss and its separate elements do not undergo essential change and the process quickly tends to a stationary regime as shown in Fig.8. With increase of the Reynolds number in case of stationary and non-stationary between occurring energy losses significant difference is developed and for becoming stationary the process takes long-time duration. To evaluate individual elements of energy losses occurring in the non-stationary flow it is necessary to plot their variation graphs and determine boundaries of their relation.

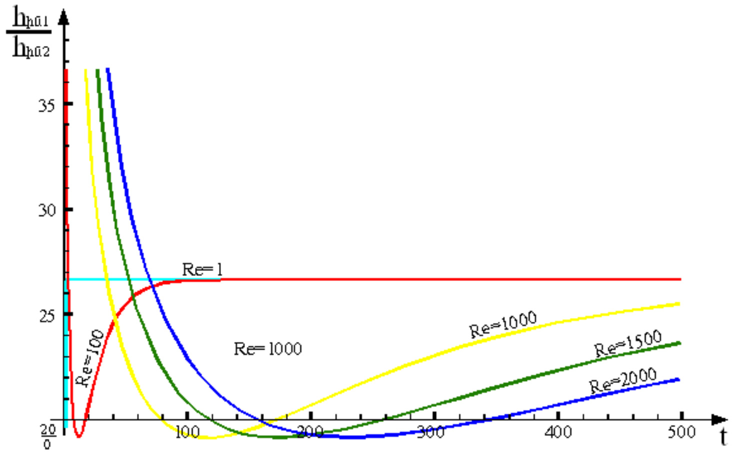

Fig. 9. Variation of relation of individual elements of energy loss when $Re = 1, 100, 1000, 1500, 2000$

Fig. 9 shows variation of inertial members relation arising due to velocity diagram change and velocity change in case of one-dimensional flow for $Re = 1, 100, 1500, 2000$ reynolds number. From the plotted graphs it follows that the values of $\frac{h_{i1}}{h_{i2}}$ relation in case of accelerating non-stationary flow in the initial moment of flow are large which means significant change of velocity gradient and uniformity of velocity field. Then occurs deformation of the velocity diagram because of which the gradient of velocity variation decreases, velocity deformation increases, and $\frac{h_{i1}}{h_{i2}}$ relation tends to 19. After that this

relation gradually increases assymptotically up to 26.8. These boundaries have been undergone computerized verification. The moment non-stationary flow starts $\frac{h_{i1}}{h_{i2}}$ relation is 57.2, then it decreases to 19.3 which with time tends to 26.1.

Fig. 10 to 13 show regularities of variation of friction stresses developed between layers of the fluid obtained in accordance with the velocity field for different values of the Reynolds number.

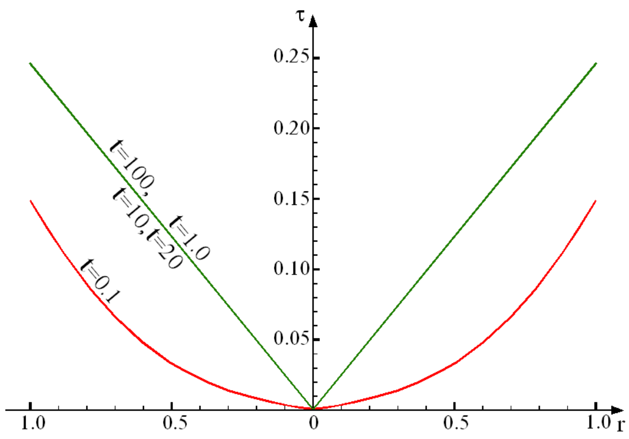

Fig. 10. Regularity of shear stresses variation when $Re = 1$, ($t_{st} = 0.824$)

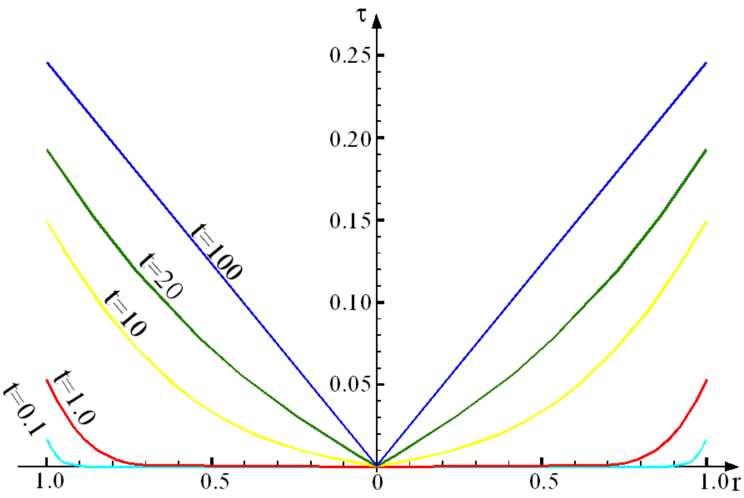

Fig. 11. Regularity of shear stresses variation when $Re = 100$, ($t_{st} = 81.407$)

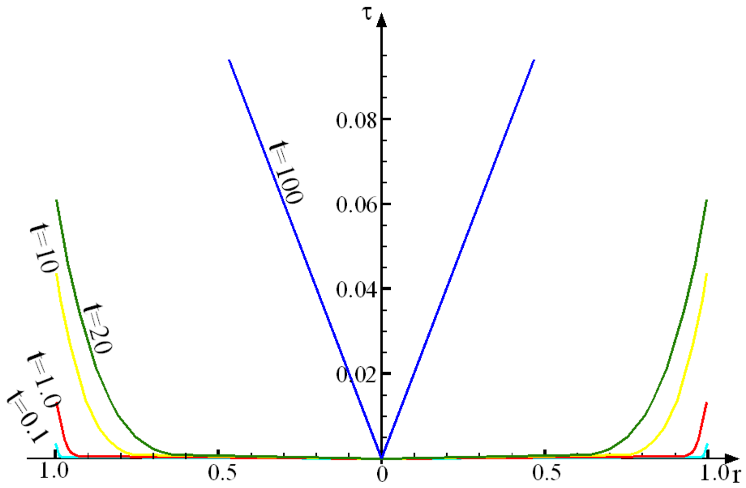

Fig. 12. Regularity of shear stresses variation when $\mathrm{Re}=1500,\ (t_{st}=1221.1)$

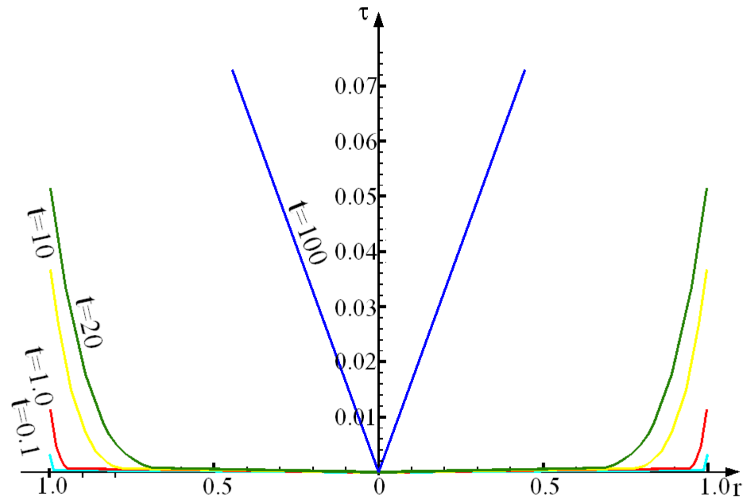

Fig. 13. Regularity of shear stresses variation when $\mathrm{Re}=2000,\ (t_{st}=1628.13)$

Equivalent to the variation of the velocity field also vary shear stresses and accordingly vary shear stresses arising near fixed wall. Therefore, energy losses will occur due to friction forces developed by friction between the fluid and fixed wall. These losses depending on the change of shear stresses similarly will have variable nature. energy losses near a fixed wall developed by shear stresses expresses by friction stresses developed near the fixed wall will [2, 27] be given as

$$h_w = \frac{\tau_0 l}{\rho g R} = \frac{4\tau_0 l}{\rho g d} \tag{27}$$

Having the regularity of variation of shear stress (24) we can derive the law of variation of energy losses developed only near-wall shear stresses. Fig.14 illustrates variation of energy losses developed by shear stresses which appear near the wall for different values of the Reynolds number. Curves have been plotted using computer technology.

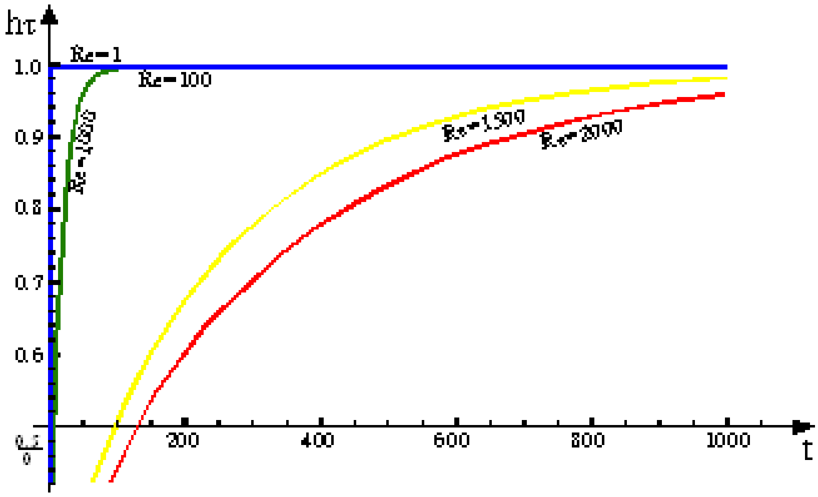

Fig. 14. Regularities of energy losses variation developed near-wall shear stresses when $Re = 1, 100, 1500, 2000$

The obtained graphs shape and character are equivalent to curves reflecting energy losses developed in non-stationary flow (see Fig.8). To make quantitative comparisons the graph of these two losses relation variation has been drawn (Fig. 15).

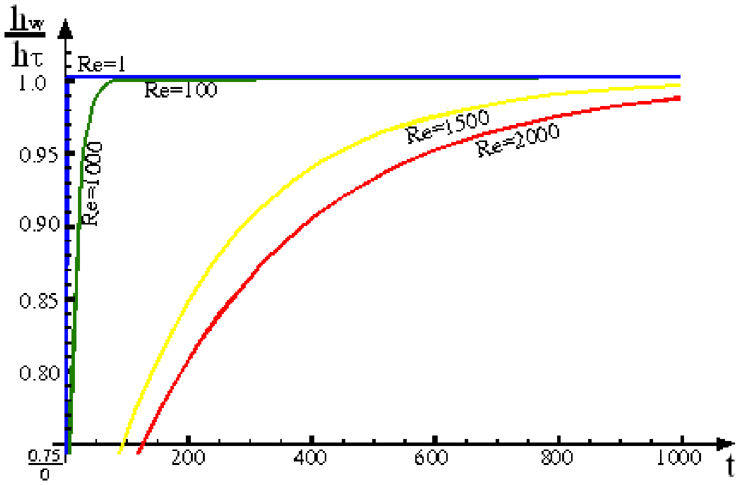

Fig. 15 Variation of energy losses relation developed by energy loss and near-wall shear stresses

Conclusions

a) **In non-stationary flow energy losses strictly differ from ones developed only friction stresses arising near fixed wall.**
b) **In case of small values of the Reynolds number practically energy losses do not differ from energy losses developed by near-wall friction stresses.**
c) **In non-stationary flow the sum of energy losses' individual elements tends to be equal to energy losses developed by friction stresses arising near the wall.**

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